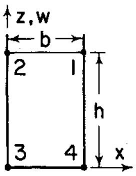

Fig. 1 One-element model for sample fluid sloshing problem.

$$F = \tfrac{1}{2}\int_{\text{vol}} \dot{\boldsymbol{\varepsilon}}^T \mathbf{C} \dot{\boldsymbol{\varepsilon}}\, dV = \tfrac{1}{2}\dot{\mathbf{q}}^T \int_{\text{vol}} \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV \dot{\mathbf{q}} \tag{6}$$

where $\mathbf{C}$ is a symmetric matrix with nonzero terms $C_{11} = C_{22} = C_{33} = 4\mu/3$, $C_{12} = C_{13} = C_{23} = -2\mu/3$ and $C_{44} = C_{55} = C_{66} = \mu$. Thus, according to Eq. (1), the integral in Eq. (6) is a damping matrix and appears on the left hand side of Eq. (4). The same result is obtained if the left-hand side of Eq. (4) is augmented by the initial-stress matrix[3]

$$\int_{\text{vol}} \mathbf{B}^T \boldsymbol{\sigma}\, dV = \int_{\text{vol}} \mathbf{B}^T \mathbf{C} \dot{\boldsymbol{\varepsilon}}\, dV = \int_{\text{vol}} \mathbf{B}^T\mathbf{C}\mathbf{B}\, dV \dot{\mathbf{q}} \tag{7}$$

As an example, consider plane motion in a rectangular tank of fluid, Fig. 1. Let the thickness be one unit and ρ constant. The fluid is modeled by a single linear element having four corner nodes.[3] The only nonzero nodal freedoms are w_1 and w_2, hence the displacement field is

$$w = (z/h)[(x/b)w_1 + (1-x/b)w_2] \tag{8}$$

Equations (4, 6, and 8) yield

$$\frac{\rho hb}{18}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} + \frac{\mu}{9hb}\begin{bmatrix} 4b^2+3h^2 & 2b^2-3h^2 \\ 2b^2-3h^2 & 4b^2+3h^2 \end{bmatrix}\begin{Bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} + \left(\frac{S}{b}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \left(\frac{\rho g b}{6} + \frac{Kb}{6h}\right)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \mathbf{Q} \tag{9}$$

For harmonic motion of an incompressible fluid, $w_1 = -w_2 = a \sin \omega t$ and $\ddot{w}_1 = -\ddot{w}_2 = -a\omega^2 \sin \omega t$. Let us set $\mu = S = \mathbf{Q} = 0$ and $h = 3b/2$. The solution of the eigenvalue problem is $\omega^2 = 3.00g/h$. For the same problem, using six elements of a different type, Ref. 2 obtains $\omega^2 = 3.46g/h$. The exact solution, Eq. (10) of Ref. 2, is $\omega^2 = 4.71g/h$.

A better fit to actual surface waves is achieved by use of the same cubic as used for beam elements. Linear edge displacements may be retained for submerged element boundaries.

References

[1] Hunt, D. A., "Discrete Element Idealization of an Incompressible Liquid for Vibration Analysis," *AIAA Journal*, Vol. 8, No. 6, June 1970, pp. 1001–1004.

[2] Hunt, D. A., "Discrete Element Structural Theory of Fluids," *AIAA Journal*, Vol. 9, No. 3, March 1971, pp. 457–461.

[3] Zienkiewicz, O. C., *The Finite Element Method in Engineering Science*, McGraw-Hill, London, 1971, pp. 16–24, 108, 326–327, 421–423.

[4] Rayleigh, J. W. S., *Theory of Sound*, Dover, New York, 1945, Vol. I, pp. 102–103, Vol. II, pp. 312–315.

Errata

Eigenvalues and Eigenvectors for Solutions to the Radiative Transport Equation

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EQUATION (23) should read

$$I(\tau, x_i) = c_1 + c_p(\tau - x_i) + \sum_{j=2}^{p/2}\left\{\frac{1-\lambda_j x_j}{1-\lambda_j^2 x_i^2}\right\} \times \{c_j(1-\lambda_j x_i)e^{\lambda_j\tau} + c_{p+1-j}(1+\lambda_j x_i)e^{-\lambda_j\tau}\} \tag{1}$$

instead of

$$I(\tau, x_i) = c_1 + c_p\tau + \sum_{j=2}^{p/2}\left\{\frac{1-\lambda_j x_j}{1-\lambda_j^2 x_i^2}\right\} \times \{c_j(1-\lambda_j x_i)e^{\lambda_j\tau} + c_{p+1-j}(1+\lambda_j x_i)e^{-\lambda_j\tau}\} \tag{2}$$

The reason for this modification is because of the change in the form of the eigenvectors associated with the repeated eigenvalue $\lambda_1 = \lambda_p = 0$. The general solution of Eq. (2), p. 974, is given by Eq. (21). However, in order to provide two linearly independent solutions for the repeated eigenvalue, Eq. (21) becomes

$$I(\tau, x_i) = (c_1 + c_p\tau)v_{i1}e^{\lambda_1\tau} + c_p v_{ip}e^{\lambda_p\tau} + \sum_{j=2}^{p\ 1} c_j v_{ij} e^{\lambda_j\tau}, \quad i = 1, \ldots, p \tag{3}$$

where the exponentials outside of the summation are equal to unity since $\lambda_1 = \lambda_p = 0$. The constants c_1 and c_p are integration constants and v_{i1} and v_{ip} are the ith components of the first and pth eigenvectors. Now for $\lambda_1 = 0$, Eq. (20) is valid for determining v_{i1}, which yields

$$v_{i1} = [(1-\lambda_1 x_1)/(1+\lambda_1 x_i)]v_{p1}, \quad i = 1, \ldots, p \tag{4}$$

Since (as shown before) v_{p1} is arbitrary, choose $v_{p1} = 1$ and then because $\lambda_1 = 0$, Eq. (4) yields

$$v_{i1} = 1, \quad i = 1, \ldots, p \tag{5}$$

Now to determine the second "independent" eigenvector associated with the repeated eigenvalue, a special procedure[1] must be used. The eigenvector for the repeated eigenvalue ($\lambda_p = 0$) must satisfy

$$\sum_{j=1}^{p} (B_{ij} - \delta_{ij}\lambda_p)v_{jp} = v_{i1} \tag{6}$$

Note v_{i1} is determined by a similar expression except the right-hand side of Eq. (6) would be zero and subscript p would be replaced by subscript 1. Observing that $v_{i1} = 1$, substituting for B_{ij} ($W = 1.0$) from Eq. (3), p. 974, and simplifying the algebra, Eq. (6) becomes

$$\tfrac{1}{2}\sum_{j=1}^{p} a_j v_{jp} = v_{ip}(\lambda_p x_i + 1) + x_i, \quad i = 1, \ldots, p \tag{7}$$

Now for the pth eigenvector, subtract the $(p+1-k)$th equation from the other $p-1$ equations. Then all the equations except the $(p+1-k)$th equation become

$$0 = [v_{ip}(\lambda_p x_i + 1) + x_i] - [(1+\lambda_p x_{p+1-k})v_{p+1-k,p} + x_{p+1-k}] \tag{8}$$

but since $\lambda_p = 0$, this yields

$$v_{ip} = v_{p+1-k,p} + x_{p+1-k} - x_i, \quad i = 1, \ldots, p \tag{9}$$

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Note that this is an identity for $i = p+1-k$. Thus Eq. (9) holds for $i = 1, \ldots, p$. Substituting Eq. (9) into Eq. (7) readily reveals that the solution for the components of the pth eigenvector given by Eq. (9) satisfy the eigenvector equation for arbitrary $v_{p+1-k,p}$. Therefore, choosing $v_{p+1-k,p} = -x_{p+1-k}$ and substituting into Eq. (9) yields

$$v_{ip} = -x_i, \quad i = 1, \ldots, p \tag{10}$$

Now substituting Eqs. (5) and (10) with $\lambda_1 = \lambda_p = 0$ into Eq. (3) yields Eq. (1), which is the correct form of Eq. (23).

Reference

[1] Kaplan, W., *Ordinary Differential Equations*, Addison-Wesley, Reading, Mass., 1958, Chap. VI, p. 288.

Optimal Desaturation of Momentum Exchange Control Systems

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IN Eq. (A4), the plus sign appearing in front of η should be changed to a minus sign (both places). The footnote at the bottom of page 20, first column, should be rewritten as: * Recall that, in the complex plane, the spectrum of A is the spectrum of $(A-\eta I), \eta \leqq 0$, shifted horizontally to the left by the amount $-\eta$.

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Plane Stress Analysis of an Annular Disk with Distorted Inner Hole

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IN the above Technical Note, the author's last name was incorrectly given as "Tim." It should have been "Kim." The editors regret the error.

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